6. The Structure Theorem for Abelian Groups

1. Find a direct sum of cyclic groups which is isomorphic to the abelian group presented by the matrix $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$.

2. Write the group generated by x, y, with the relation $3x + 4y = 0$ as a direct sum of cyclic groups.

3. Find an isomorphic direct product of cyclic groups, when V is the abelian group generated by x, y, z, with the given relations.

(a) $3x + 2y + 8z = 0,\ 2x + 4z = 0$

(b) $x + y = 0,\ 2x = 0,\ 4x + 2z = 0,\ 4x + 2y + 2z = 0$

(c) $2x + y = 0,\ x - y + 3z = 0$

(d) $2x - 4y = 0, \quad 2x + 2y + z = 0$

(e) $7x + 5y + 2z = 0,\ 3x + 3y = 0,\ 13x + 11y + 2z = 0$

4. Determine the number of isomorphism classes of abelian groups of order 400.

5. Classify finitely generated modules over each ring.

(a) $\mathbb{Z}/(4)$ **(b)** $\mathbb{Z}/(6)$ **(c)** $\mathbb{Z}/n\mathbb{Z}$.

6. Let R be a ring, and let V be an R-module, presented by a diagonal $m \times n$ matrix A: $V \approx R^m/AR^n$. Let $(v_1, \ldots, v_m)$ be the corresponding generators of V, and let d_i be the diagonal entries of A. Prove that V is isomorphic to a direct product of the modules $R/(d_i)$.

7. Let V be the $\mathbb{Z}[i]$-module generated by elements v_1, v_2 with relations $(1 + i)v_1 + (2 - i)v_2 = 0, \quad 3v_1 + 5iv_2 = 0$. Write this module as a direct sum of cyclic modules.

8. Let $W_1, \ldots, W_k$ be submodules of an R-module V such that $V = \Sigma W_i$. Assume that $W_1 \cap W_2 = 0,\ (W_1 + W_2) \cap W_3 = 0, \ldots, (W_1 + W_2 + \cdots + W_{k-1}) \cap W_k = 0$. Prove that V is the direct sum of the modules $W_1, \ldots, W_k$.

* **9.** Prove the following.

(a) The number of elements of $\mathbb{Z}/(p^e)$ whose order divides p^ν is p^ν if $\nu \leq e$, and is p^e if $\nu \geq e$.

(b) Let $W_1, \ldots, W_k$ be finite abelian groups, and let u_j denote the number of elements of W_j whose order divides a given integer q. Then the number of elements of the product group $V = W_1 \times \cdots \times W_k$ whose order divides q is $u_1 \cdots u_k$.

(c) With the above notation, assume that W_j is a cyclic group of prime power order $d_j = p^{e_j}$. Let r_1 be the number of d_j equal to a given prime p, let r_2 be the number of d_j equal to p^2, and so on. Then the number of elements of V whose order divides p^ν is p^{s_ν}, where $s_1 = r_1 + \cdots + r_k$, $s_2 = r_1 + 2r_2 + \cdots + 2r_k$, $s_3 = r_1 + 2r_2 + 3r_3 + \cdots + 3r_k$, and so on.

(d) Theorem (6.9).

7. Application to Linear Operators

1. Let T be a linear operator whose matrix is $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Is the corresponding $\mathbb{C}[t]$-module cyclic?